Balancing Communication and Computations in Gradient Tracking Methods for Decentralized Optimization

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Problem

$$\min_{x\in\mathbb{R}^d} f(x) = \sum_{i=1}^{n} f_i(x)$$

Each function f_i is only known to agent i $\forall$ $i = 1, 2, ..., n$

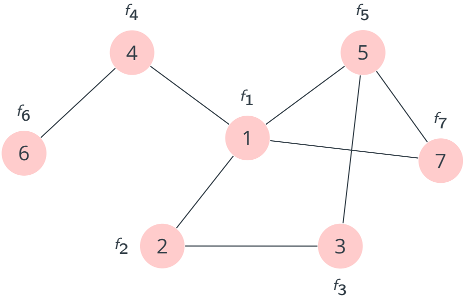

Figure: Distributed Network Example

Problem

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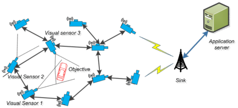
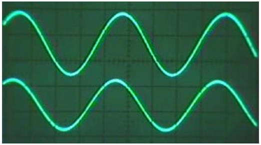


(a) Sensor Networks

You et al. 2013

(b) Machine Learning

Tom Taulli, Forbes 2019



(c) Signal Processing

Signal Processing, MIT OCW 2011

Consensus Optimization Problem

$$\min_{x_i \in \mathbb{R}^d} \sum_{i=1}^{n} f_i(x_i)$$

$$s.t. \quad x_i = x_j \quad \forall\, i, j \in \mathcal{E}$$

Each node keeps a local copy $x_i \ \forall\ i = 1, 2, ..., n$

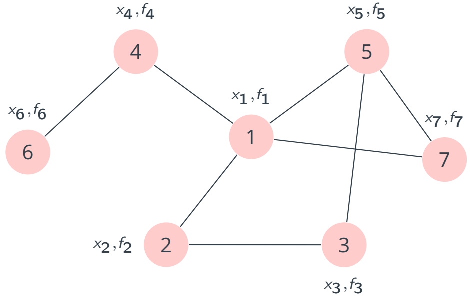


Figure: Distributed Network Example

Consensus Optimization Problem

$$\min_{x_i \in \mathbb{R}^d} f(\mathbf{x}) = \sum_{i=1}^{n} f_i(x_i)$$
$$s.t. \quad (\mathbf{W} \otimes I_d)\mathbf{x} = \mathbf{x}$$

- $\mathbf{x}$ is a concatenation of all local x_i's
- $\mathbf{W}$ is a symmetric doubly-stochastic matrix that defines the connections in the network

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^{nd}, \quad \mathbf{W} = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & \cdots & w_{nn} \end{bmatrix} \in \mathbb{R}^{n \times n}$$

Consensus Optimization Problem

$$\min_{x_i \in \mathbb{R}^d} f(\mathbf{x}) = \sum_{i=1}^{n} f_i(x_i)$$
$$s.t. \quad \mathbf{Z}\,\mathbf{x} = \mathbf{x}$$

- $\mathbf{x}$ is a concatenation of all local x_i's
- $\mathbf{W}$ is a symmetric doubly-stochastic matrix that defines the connections in the network

$$\mathbf{Z} = \mathbf{W} \otimes I_d \in \mathbb{R}^{nd \times nd}$$

Literature Review

1. Sublinearly Converging Methods

   DGD [Bertsekas, Tsitsiklis, et al. 1989, Nedic and Ozdaglar 2009, Sundhar Ram et al. 2010, Tsianos et al. 2012], **NN** [Mokhtari et al. 2017], **NEAR-DGD** [Berahas et al. 2018], ...

2. Linearly Converging Methods

   Push-pull [Pu, Shi, et al. 2020], **DIGing** [Nedic, Olshevsky, et al. 2017], **EXTRA** [Shi et al. 2015], **SONATA** [Sun et al. 2022], **NEXT** [Di Lorenzo and Scutari 2015], **Aug-DGM** [Xu et al. 2015], **LU-GT** [Nguyen et al. 2022], ...

3. Asynchronous Methods

   [Bertsekas, Tsitsiklis, et al. 1989], [Ram, Veeravalli, and Nedic 2009], **HOGWILD** [Recht et al. 2011], [Wei and Ozdaglar 2013], ...

4. Stochastic Algorithms

   DSGT and **GSGT** [Pu and Nedić 2021], **ProxiSkip** [Mishchenko et al. 2022], ...

Gradient Tracking Methods

Push-pull [Pu, Shi, et al. 2020], **DIGing** [Nedic, Olshevsky, et al. 2017], **EXTRA** [Shi et al. 2015], **SONATA** [Sun et al. 2022], **NEXT** [Di Lorenzo and Scutari 2015], **Aug-DGM** [Xu et al. 2015], ...

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{Z}\,\mathbf{x}_k - \alpha \mathbf{y}_k, \\ \mathbf{y}_{k+1} &= \mathbf{Z}\,\mathbf{y}_k + \nabla \mathbf{f}(\mathbf{x}_{k+1}) - \nabla \mathbf{f}(\mathbf{x}_k) \end{aligned}$$

$$\mathbf{x}_k = \begin{bmatrix} x_{1,k} \\ x_{2,k} \\ \vdots \\ x_{n,k} \end{bmatrix} \in \mathbb{R}^{nd}, \quad \mathbf{y}_k = \begin{bmatrix} y_{1,k} \\ y_{2,k} \\ \vdots \\ y_{n,k} \end{bmatrix} \in \mathbb{R}^{nd}, \quad \nabla \mathbf{f}(\mathbf{x}_k) = \begin{bmatrix} \nabla f_1(x_{1,k}) \\ \nabla f_2(x_{2,k}) \\ \vdots \\ \nabla f_n(x_{n,k}) \end{bmatrix} \in \mathbb{R}^{nd}$$

- Use an additional dual variable $\mathbf{y}_k$ to track the gradient
- Constant α : Linear converge to solution

Gradient Tracking Methods

Push-pull [Pu, Shi, et al. 2020], **DIGing** [Nedic, Olshevsky, et al. 2017], **EXTRA** [Shi et al. 2015], **SONATA** [Sun et al. 2022], **NEXT** [Di Lorenzo and Scutari 2015], **Aug-DGM** [Xu et al. 2015], ...

DIGing:

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{Z}\,\mathbf{x}_k - \alpha \mathbf{y}_k, \\ \mathbf{y}_{k+1} &= \mathbf{Z}\,\mathbf{y}_k + \nabla \mathbf{f}(\mathbf{x}_{k+1}) - \nabla \mathbf{f}(\mathbf{x}_k) \end{aligned}$$

Aug-DGM:

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{Z}(\mathbf{x}_k - \alpha \mathbf{y}_k), \\ \mathbf{y}_{k+1} &= \mathbf{Z}(\mathbf{y}_k + \nabla \mathbf{f}(\mathbf{x}_{k+1}) - \nabla \mathbf{f}(\mathbf{x}_k)) \end{aligned}$$

Gradient Tracking Methods

Push-pull [Pu, Shi, et al. 2020], **DIGing** [Nedic, Olshevsky, et al. 2017], **EXTRA** [Shi et al. 2015], **SONATA** [Sun et al. 2022], **NEXT** [Di Lorenzo and Scutari 2015], **Aug-DGM** [Xu et al. 2015], ...

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Aug-DGM:

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- Choice of information shared affects both convergence and practical implementation

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DIGing:

$$\mathbf{x}_{k+1} = \mathbf{Z}\mathbf{x}_k - \alpha \mathbf{y}_k,$$
$$\mathbf{y}_{k+1} = \mathbf{Z}\mathbf{y}_k + \nabla \mathbf{f}(\mathbf{x}_{k+1}) - \nabla \mathbf{f}(\mathbf{x}_k)$$

Aug-DGM:

$$\mathbf{x}_{k+1} = \mathbf{Z}(\mathbf{x}_k - \alpha \mathbf{y}_k),$$
$$\mathbf{y}_{k+1} = \mathbf{Z}(\mathbf{y}_k + \nabla \mathbf{f}(\mathbf{x}_{k+1}) - \nabla \mathbf{f}(\mathbf{x}_k))$$

- Choice of information shared affects both convergence and practical implementation
- Applications require a different composition communication and computation steps to achieve overall efficiency

This Talk

1. We develop a gradient tracking algorithmic framework (`GTA`) to unify gradient tracking methods.
2. Provide flexibility in number of communication and computation steps in each iteration in `GTA`.
3. Provide sufficient conditions for linear rate of convergence.
4. Illustrate benefits of this flexibility with numerical experiments.

GTA Framework

$\mathbf{W} \in \mathbb{R}^{n \times n} \rightarrow$ mixing matrix

- Symmetric, Doubly Stochastic
- Represents the network, i.e., $w_{ii} > 0$ and $w_{ij} > 0$ iff $(i, j) \in \mathcal{E}$
- $\left\| \mathbf{W} - \frac{1_n 1_n^T}{n} \right\|_2 = \beta \in [0, 1)$

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- $\left\| \mathbf{W} - \frac{1_n 1_n^T}{n} \right\|_2 = \beta \in [0, 1)$

$\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3, \mathbf{W}_4 \in \mathbb{R}^{n \times n} \rightarrow$ communication matrices

- Symmetric, Doubly Stochastic
- Represents a subset of edges of the network, i.e., $w_{1,ii} > 0$ and $w_{1,ij} \geq 0$ if $(i, j) \in \mathcal{E}$ else $w_{1,ij} = 0$
- $\left\| \mathbf{W}_i - \frac{1_n 1_n^T}{n} \right\|_2 = \beta_i \in [0, 1] \quad \forall \quad i = 1, 2, 3, 4$

GTA Framework

$\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3, \mathbf{W}_4 \rightarrow$ communication matrices

Single communication and computation step in each iteration.

$$\begin{aligned}
\mathbf{x}_{k+1} &= \mathbf{Z}_1 \mathbf{x}_k - \alpha \mathbf{Z}_2 \mathbf{y}_k, \\
\mathbf{y}_{k+1} &= \mathbf{Z}_3 \mathbf{y}_k + \mathbf{Z}_4 (\nabla \mathbf{f}(\mathbf{x}_{k+1}) - \nabla \mathbf{f}(\mathbf{x}_k))
\end{aligned}$$

where $\mathbf{Z}_i = \mathbf{W}_i \otimes I_d \in \mathbb{R}^{nd \times nd} \quad \forall \quad i = 1, 2, 3, 4$

GTA Framework Special Cases

Mixing matrix $\mathbf{W}$ and $\mathbf{Z} = \mathbf{W} \otimes I_d$

GTA-1 (DIGing, EXTRA, ...)

$$
\begin{aligned}
\mathbf{x}_{k+1} &= \mathbf{Z}\mathbf{x}_k - \alpha \mathbf{y}_k \\
\mathbf{y}_{k+1} &= \mathbf{Z}\mathbf{y}_k + \nabla \mathbf{f}(\mathbf{x}_{k+1}) - \nabla \mathbf{f}(\mathbf{x}_k)
\end{aligned}
$$

GTA-2 (NEXT, SONATA, ...)

$$
\begin{aligned}
\mathbf{x}_{k+1} &= \mathbf{Z}\left(\mathbf{x}_k - \alpha \mathbf{y}_k\right) \\
\mathbf{y}_{k+1} &= \mathbf{Z}\mathbf{y}_k + \nabla \mathbf{f}(\mathbf{x}_{k+1}) - \nabla \mathbf{f}(\mathbf{x}_k)
\end{aligned}
$$

GTA-3 (Aug-DGM, ATC-DIGing, ...)

$$
\begin{aligned}
\mathbf{x}_{k+1} &= \mathbf{Z}\left(\mathbf{x}_k - \alpha \mathbf{y}_k\right) \\
\mathbf{y}_{k+1} &= \mathbf{Z}\left(\mathbf{y}_k + \nabla \mathbf{f}(\mathbf{x}_{k+1}) - \nabla \mathbf{f}(\mathbf{x}_k)\right)
\end{aligned}
$$

GTA Framework - Convergence Analysis

Definitions

$$\bar{x}_k = \frac{1}{n}\sum_{i=1}^{n} x_{i,k}, \quad \bar{y}_k = \frac{1}{n}\sum_{i=1}^{n} y_{i,k}$$

GTA Framework - Convergence Analysis

Definitions

$$\bar{x}_k = \frac{1}{n}\sum_{i=1}^{n} x_{i,k}, \quad \bar{y}_k = \frac{1}{n}\sum_{i=1}^{n} y_{i,k}$$

$$r_k = \begin{bmatrix} \|\bar{x}_k - x^*\|_2 \\ \|\mathbf{x}_k - \bar{\mathbf{x}}_k\|_2 \\ \|\mathbf{y}_k - \bar{\mathbf{y}}_k\|_2 \end{bmatrix}, \quad \bar{\mathbf{x}}_k = \begin{bmatrix} \bar{x}_k \\ \bar{x}_k \\ \vdots \\ \bar{x}_k \end{bmatrix} \in \mathbb{R}^{nd}, \quad \bar{\mathbf{y}}_k = \begin{bmatrix} \bar{y}_k \\ \bar{y}_k \\ \vdots \\ \bar{y}_k \end{bmatrix} \in \mathbb{R}^{nd}$$

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Assumption

1. The function f is $\mu > 0$ strongly convex and each component function f_i has $L > 0$ Lipschitz continuous gradients.

GTA Framework - Step size condition

$$\mathbf{x}_{k+1} = \mathbf{Z}_1\mathbf{x}_k - \alpha\mathbf{Z}_2\mathbf{y}_k,$$
$$\mathbf{y}_{k+1} = \mathbf{Z}_3\mathbf{y}_k + \mathbf{Z}_4(\nabla\mathbf{f}(\mathbf{x}_{k+1}) - \nabla\mathbf{f}(\mathbf{x}_k))$$

Theorem

Suppose Assumption 1 holds and $\beta_1, \beta_3 < 1$ in GTA Framework, then $\|r_k\|_2$ goes to 0 at a linear rate if

$$\alpha < \min\left\{\frac{1}{L}, \frac{1-\beta_3}{L\beta_4}, \frac{(1-\beta_1+2\beta_2)}{2\beta_2\kappa(L+\mu)}\left(\sqrt{1+\frac{4(1-\beta_1)(1-\beta_3)\beta_2(\kappa+1)}{\beta_4(1-\beta_1+2\beta_2)^2}}-1\right)\right\}$$

where $\kappa = \frac{L}{\mu}$.

GTA Framework Cases - Step size condition

Theorem

Suppose Assumption 1 holds, $\|r_k\|_2$ goes to 0 at a linear rate for the special cases if

$$\text{GTA-1:}\quad \alpha < \min\left\{\frac{1-\beta}{L}, \frac{(3-\beta)}{2\kappa(L+\mu)}\left(\sqrt{1+4(\kappa+1)\left(\frac{1-\beta}{3-\beta}\right)^2}-1\right)\right\}$$

$$\text{GTA-2:}\quad \alpha < \min\left\{\frac{1-\beta}{L}, \frac{(1+\beta)}{2\kappa(L+\mu)\beta}\left(\sqrt{1+4(\kappa+1)\beta\left(\frac{1-\beta}{1+\beta}\right)^2}-1\right)\right\}$$

$$\text{GTA-3:}\quad \alpha < \min\left\{\frac{1}{L}, \frac{1-\beta}{L\beta}, \frac{(1+\beta)}{2\kappa(L+\mu)\beta}\left(\sqrt{1+4(\kappa+1)\left(\frac{1-\beta}{1+\beta}\right)^2}-1\right)\right\}$$

where $\kappa = \frac{L}{\mu}$.

GTA Framework Cases - Rate of Convergence

Theorem

Suppose Assumption 1 holds and $\alpha \leq \frac{1}{L}$, $\|r_k\|_2$ goes to 0 at a linear rate upper bounded by the following expressions

$$\textit{GTA-1:} \quad \max\left\{1 - \frac{\alpha\mu}{2},\ \beta + \sqrt{\alpha L}\left(2.5 + \sqrt{\kappa}\right)\right\}$$

$$\textit{GTA-2:} \quad \max\left\{1 - \frac{\alpha\mu}{2},\ \beta + \sqrt{\alpha L}\left(2.5 + \sqrt{\kappa\beta}\right)\right\}$$

$$\textit{GTA-3:} \quad \max\left\{1 - \frac{\alpha\mu}{2},\ \beta\left(1 + \sqrt{\alpha L}\left(2.5 + \sqrt{\kappa}\right)\right)\right\}$$

where $\kappa = \frac{L}{\mu}$.

GTA Framework - Numerical Experiments

Almost Full network $\beta = 0.25$

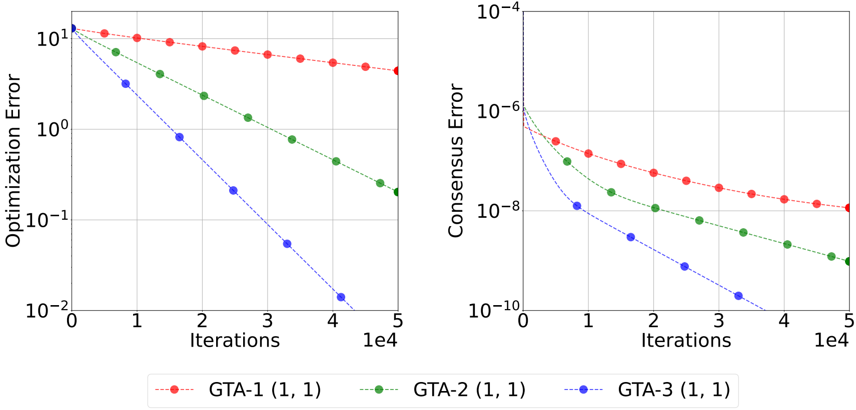


Figure: Quadratics, $n = 16$, $d = 10$, $\kappa = 10^4$

GTA Framework - Numerical Experiments

Cyclic Network $\beta = 0.992$

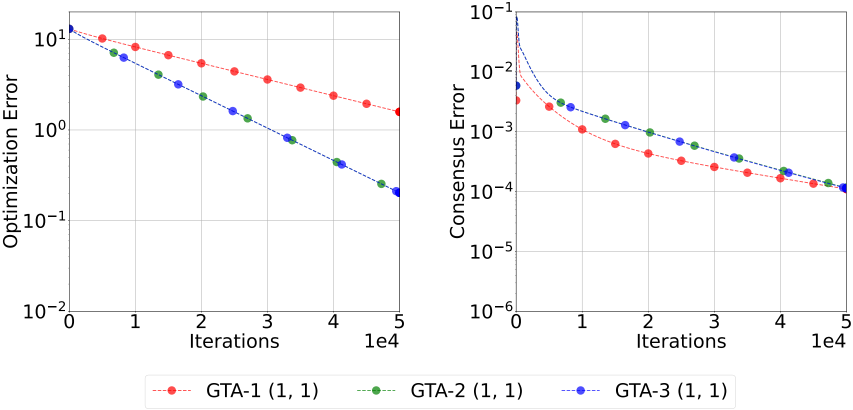
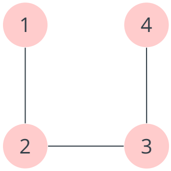


Figure: Quadratics, $n = 16$, $d = 10$, $\kappa = 10^4$

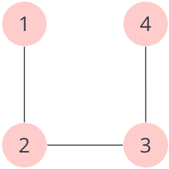
Multiple Communications



(a) Single Communication

$$\mathbf{W} = \begin{bmatrix} 0.9 & 0.1 & 0 & 0 \\ 0.1 & 0.8 & 0.1 & 0 \\ 0 & 0.1 & 0.8 & 0.1 \\ 0 & 0 & 0.1 & 0.9 \end{bmatrix}$$

Multiple Communications



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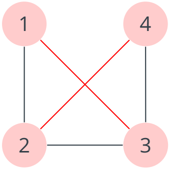
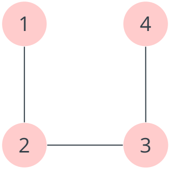


(b) 2 Communications

$$\mathbf{W} = \begin{bmatrix} 0.9 & 0.1 & 0 & 0 \\ 0.1 & 0.8 & 0.1 & 0 \\ 0 & 0.1 & 0.8 & 0.1 \\ 0 & 0 & 0.1 & 0.9 \end{bmatrix}$$

Multiple Communications



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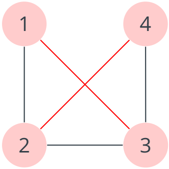
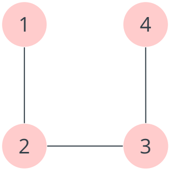


(b) 2 Communications

$$\mathbf{W} = \begin{bmatrix} 0.9 & 0.1 & 0 & 0 \\ 0.1 & 0.8 & 0.1 & 0 \\ 0 & 0.1 & 0.8 & 0.1 \\ 0 & 0 & 0.1 & 0.9 \end{bmatrix} \qquad \mathbf{W}^2 = \begin{bmatrix} 0.82 & 0.17 & 0.01 & 0 \\ 0.17 & 0.66 & 0.16 & 0.01 \\ 0.01 & 0.16 & 0.66 & 0.17 \\ 0 & 0.01 & 0.17 & 0.82 \end{bmatrix}$$

Multiple Communications



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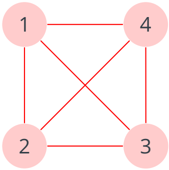
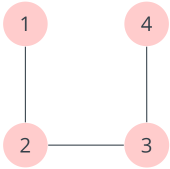
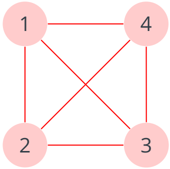


(b) 200 Communications

Multiple Communications



(a) Single Communication



(b) 200 Communications

$$\mathbf{W} = \begin{bmatrix} 0.9 & 0.1 & 0 & 0 \\ 0.1 & 0.8 & 0.1 & 0 \\ 0 & 0.1 & 0.8 & 0.1 \\ 0 & 0 & 0.1 & 0.9 \end{bmatrix}$$

$$\mathbf{W}^{200} = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix}$$

GTA Framework - Multiple Communications

$n_c \rightarrow$ #of communication steps

$\mathbf{W}_i \rightarrow \mathbf{W}_i^{n_c} \qquad \forall i = 1, 2, 3, 4$

$\beta_i \rightarrow \beta_i^{n_c} \qquad \forall i = 1, 2, 3, 4$

$\mathbf{Z}_i \rightarrow \mathbf{Z}_i^{n_c} = \mathbf{W}_i^{n_c} \otimes I_d \qquad \forall i = 1, 2, 3, 4$

GTA Framework - Multiple Communications

$n_c \rightarrow$ #of communication steps

$\mathbf{W}_i \rightarrow \mathbf{W}_i^{n_c} \qquad \forall i = 1,2,3,4$

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$$\mathbf{x}_{k+1} = \mathbf{Z}_1^{n_c}\mathbf{x}_k - \alpha\mathbf{Z}_2^{n_c}\mathbf{y}_k,$$
$$\mathbf{y}_{k+1} = \mathbf{Z}_3^{n_c}\mathbf{y}_k + \mathbf{Z}_4^{n_c}(\nabla\mathbf{f}(\mathbf{x}_{k+1}) - \nabla\mathbf{f}(\mathbf{x}_k))$$

GTA Framework - Multiple Communications

$$
\begin{aligned}
&n_c \to \#\text{of communication steps} \\
&\mathbf{W}_i \to \mathbf{W}_i^{n_c} && \forall i = 1,2,3,4 \\
&\beta_i \to \beta_i^{n_c} && \forall i = 1,2,3,4 \\
&\mathbf{Z}_i \to \mathbf{Z}_i^{n_c} = \mathbf{W}_i^{n_c} \otimes I_d && \forall i = 1,2,3,4
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{x}_{k+1} &= \mathbf{Z}_1^{n_c}\mathbf{x}_k - \alpha \mathbf{Z}_2^{n_c}\mathbf{y}_k, \\
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\end{aligned}
$$

With more communcation, i.e., increase in n_c

- The step size condition increases
- The rate of convergence decreases

GTA Special Cases - Multiple Communications

Mixing matrix $\mathbf{W}$ and $\mathbf{Z}^{n_c} = \mathbf{W}^{n_c} \otimes I_d$

GTA-1

$$\mathbf{x}_{k+1} = \mathbf{Z}^{n_c}\mathbf{x}_k - \alpha\mathbf{y}_k$$
$$\mathbf{y}_{k+1} = \mathbf{Z}^{n_c}\mathbf{y}_k + \nabla\mathbf{f}(\mathbf{x}_{k+1}) - \nabla\mathbf{f}(\mathbf{x}_k)$$

GTA-2

$$\mathbf{x}_{k+1} = \mathbf{Z}^{n_c}\left(\mathbf{x}_k - \alpha\mathbf{y}_k\right)$$
$$\mathbf{y}_{k+1} = \mathbf{Z}^{n_c}\mathbf{y}_k + \nabla\mathbf{f}(\mathbf{x}_{k+1}) - \nabla\mathbf{f}(\mathbf{x}_k)$$

GTA-3

$$\mathbf{x}_{k+1} = \mathbf{Z}^{n_c}\left(\mathbf{x}_k - \alpha\mathbf{y}_k\right)$$
$$\mathbf{y}_{k+1} = \mathbf{Z}^{n_c}\left(\mathbf{y}_k + \nabla\mathbf{f}(\mathbf{x}_{k+1}) - \nabla\mathbf{f}(\mathbf{x}_k)\right)$$

GTA Multiple Communications - Rate of Convergence

Theorem

Suppose Assumption 1 holds, number of communications is at least 1 ($n_c \geq 1$) and $\alpha \leq \frac{1}{L}$, $\|r_k\|_2$ goes to 0 at a linear rate upper bounded by the following expressions

For GTA-1 $$\max \left\{1 - \frac{\alpha\mu}{2},\ \beta^{n_c} + \sqrt{\alpha L}\left(2.5 + \sqrt{\kappa}\right)\right\}$$

For GTA-2 $$\max \left\{1 - \frac{\alpha\mu}{2},\ \beta^{n_c} + \sqrt{\alpha L}\left(2.5 + \sqrt{\kappa\beta^{n_c}}\right)\right\}$$

For GTA-3 $$\max \left\{1 - \frac{\alpha\mu}{2},\ \beta^{n_c}\left(1 + \sqrt{\alpha L}\left(2.5 + \sqrt{\kappa}\right)\right)\right\}$$

where $\kappa = \frac{L}{\mu}$.

GTA Multiple Communications - Numerical Experiments

Cyclic Network $\beta = 0.992$

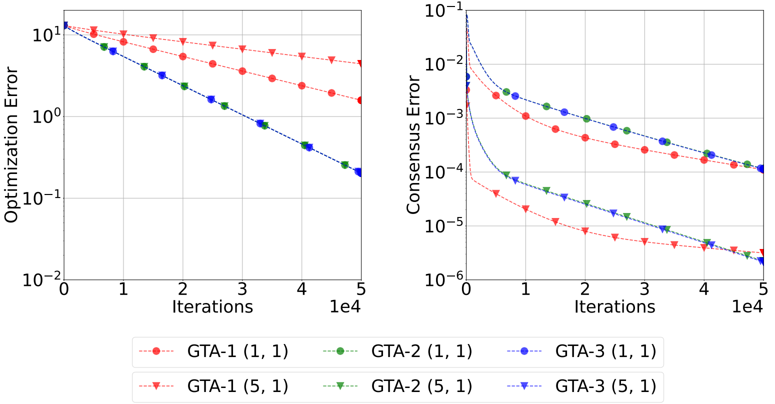

Figure: Quadratics, $n = 16$, $d = 10$, $\kappa = 10^4$

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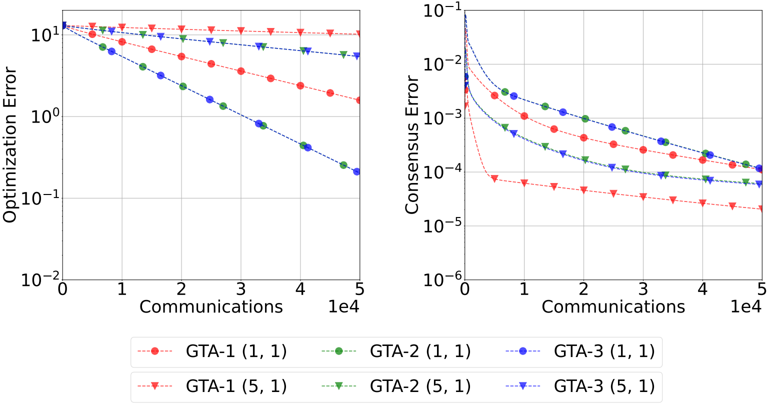

Figure: Quadratics, $n = 16$, $d = 10$, $\kappa = 10^4$

GTA - Multiple Communications and Computations

$n_c \rightarrow$ # of communication steps
$n_g \rightarrow$ # of computation steps

$$\mathbf{x}_{k+1,1} = \mathbf{Z}_1^{n_c}\mathbf{x}_{k,n_g} - \alpha\mathbf{Z}_2^{n_c}\mathbf{y}_{k,n_g}$$
$$\mathbf{y}_{k+1,1} = \mathbf{Z}_3^{n_c}\mathbf{y}_{k,n_g} + \mathbf{Z}_4^{n_c}(\nabla\mathbf{f}(\mathbf{x}_{k+1,1}) - \nabla\mathbf{f}(\mathbf{x}_{k,n_g}))$$

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For $j \rightarrow 1, 2, ..., n_g - 1$

$$\mathbf{x}_{k+1,j+1} = \mathbf{x}_{k+1,j} - \alpha\mathbf{y}_{k+1,j},$$

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GTA - Multiple Communications and Computations

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$n_g \rightarrow \#$ of computation steps

Theorem

Under previous assumptions, $\beta_1, \beta_3 < 1$, number of communication steps is at least one ($n_c \geq 1$) and number of computation steps is finite ($1 \leq n_g < \infty$), then $\exists\ \alpha > 0$, s.t. $\|r_k\|_2$ goes to 0 at a linear rate.

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Under previous assumptions, $\beta_1, \beta_3 < 1$, number of communication steps is at least one ($n_c \geq 1$) and number of computation steps is finite ($1 \leq n_g < \infty$), then $\exists\ \alpha > 0$, s.t. $\|r_k\|_2$ goes to 0 at a linear rate.

- The step size increases with an increase in n_c, i.e., number of communication steps.
- The step size is inversely proportional to n_g, i.e., number of computation steps.

GTA Special Cases

GTA-1

$$\mathbf{x}_{k+1,1} = \mathbf{Z}^{n_c}\mathbf{x}_{k,n_g} - \alpha\mathbf{y}_{k,n_g}$$
$$\mathbf{y}_{k+1,1} = \mathbf{Z}^{n_c}\mathbf{y}_{k,n_g} + \nabla\mathbf{f}(\mathbf{x}_{k+1,1}) - \nabla\mathbf{f}(\mathbf{x}_{k,n_g})$$
$\rightarrow n_g - 1$ compute steps

GTA-2

$$\mathbf{x}_{k+1,1} = \mathbf{Z}^{n_c}\left(\mathbf{x}_{k,n_g} - \alpha\mathbf{y}_{k,n_g}\right)$$
$$\mathbf{y}_{k+1,1} = \mathbf{Z}^{n_c}\mathbf{y}_{k,n_g} + \nabla\mathbf{f}(\mathbf{x}_{k+1,1}) - \nabla\mathbf{f}(\mathbf{x}_{k,n_g})$$
$\rightarrow n_g - 1$ compute steps

GTA-3

$$\mathbf{x}_{k+1,1} = \mathbf{Z}^{n_c}\left(\mathbf{x}_{k,n_g} - \alpha\mathbf{y}_{k,n_g}\right)$$
$$\mathbf{y}_{k+1,1} = \mathbf{Z}^{n_c}\left(\mathbf{y}_{k,n_g} + \nabla\mathbf{f}(\mathbf{x}_{k+1,1}) - \nabla\mathbf{f}(\mathbf{x}_{k,n_g})\right)$$
$\rightarrow n_g - 1$ compute steps

GTA Special Cases

$n_c \rightarrow \#$ of communication steps
$n_g \rightarrow \#$ of computation steps

If the same step size is employed in all three methods, their convergence rates can be ordered as:

$$\textit{GTA-3}(n_c, n_g) \leq \textit{GTA-2}(n_c, n_g) \leq \textit{GTA-1}(n_c, n_g)$$

GTA Multiple Communications and Computations - Numerical Experiments

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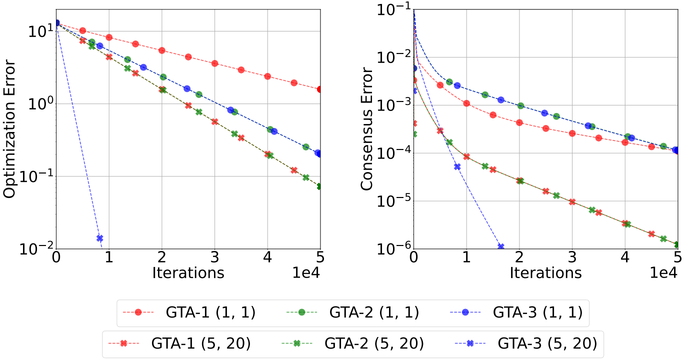

Figure: Quadratics, $n = 16$, $d = 10$, $\kappa = 10^4$

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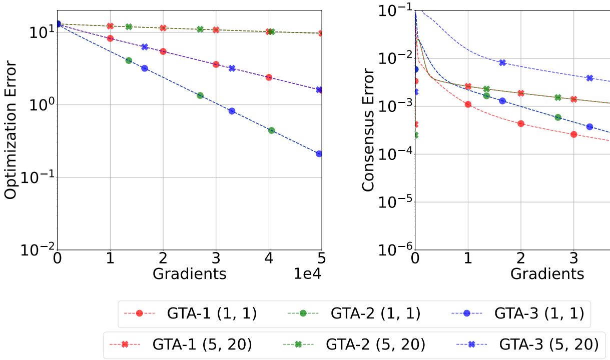

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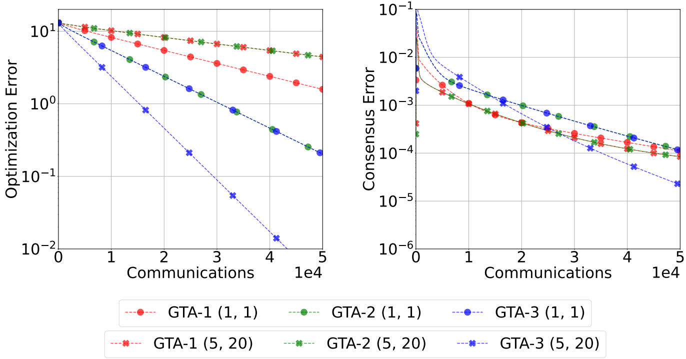

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Flexible Randomized Gradient Tracking Algorithm

Performing communications less often randomly

FedAvg [McMahan et al. 2017], **FedLin** [Mitra et al. 2021], **Scaffold** [Karimireddy et al. 2020], **Scaffnew** [Mishchenko et al. 2022], ...

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With probability p:

$$\mathbf{x}_{k+1} = \mathbf{Z}_1^{n_c}\mathbf{x}_k - \alpha\mathbf{Z}_2^{n_c}\mathbf{y}_k,$$
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Else:

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- Less rigid, achieves the desired balance in expectation
- Good theoretical and empirical performance
- Paper soon to follow !!

Conclusions

1. We provide a unifying gradient tracking algorithmic framework that allows performing theoretical comparisons between different gradient tracking methods.

2. We provide the flexibility to perform any composition of communication and computation steps in each iteration and show linear rate of convergence.

3. Adapting your algorithm to the system with this flexibility can allow you to improve convergence rate.

Paper available at : `https://arxiv.org/abs/2303.14289`

Thank You!

Questions?

Backup Slides

$$r_{k+1} \leq A(n_c) r_k$$

$$A(n_c) = \begin{bmatrix} 1 - \alpha\mu & \frac{\alpha L}{\sqrt{n}} & 0 \\ 0 & \beta_1^{n_c} & \alpha\beta_2^{n_c} \\ \sqrt{n}\alpha\beta_4^{n_c} L^2 & \beta_4^{n_c} L(\|\mathbf{Z}_1^{n_c} - I_{nd}\|_2 + \alpha L) & \beta_3^{n_c} + \alpha\beta_4^{n_c} L \end{bmatrix}$$

$$r_{k+1} \le B(n_c, n_g) r_k,$$

$$\text{where } \; B(n_c, n_g) = A(n_c, n_g) + \alpha L (n_g - 1) E(n_c, n_g)$$

$$A(n_c, n_g) = \begin{bmatrix} (1-\alpha\mu)^{n_g} & \frac{\kappa}{\sqrt{n}}(1-(1-\alpha\mu)^{n_g}) & 0 \\ 0 & \beta_1^{n_c} & \alpha\left((n_g-1)\beta_1^{n_c} + \beta_2^{n_c}\right) \\ \sqrt{n}\alpha\beta_4^{n_c} L^2 & \beta_4^{n_c} L(\|\mathbf{Z}_1^{n_c} - I_{nd}\|_2 + \alpha L) & \beta_3^{n_c} + \alpha\beta_4^{n_c} L \end{bmatrix}$$

$$E(n_c, n_g) = \begin{bmatrix} \alpha L n_g & \frac{\alpha L n_g}{\sqrt{n}} & \frac{\alpha n_g}{\sqrt{n}} \\ \sqrt{n}\alpha L \delta_1(n_c, n_g) & \alpha L \delta_1(n_c, n_g) & \alpha \delta_1(n_c, n_g) \\ \sqrt{n} L \delta_2(n_c, n_g) & L\delta_2(n_c, n_g) & \delta_2(n_c, n_g) \end{bmatrix}$$

and

$$\delta_1(n_c, n_g) = 2\beta_2^{n_c} + \beta_1^{n_c}(n_g - 2),$$

$$\delta_2(n_c, n_g) = 2\left(\beta_4^{n_c}\|\mathbf{Z}_1^{n_c} - I_{nd}\|_2 + \frac{\beta_4^{n_c}}{n_g} + \beta_3^{n_c}\right).$$